Power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Radius of convergence

If the power series $\sum a_n x^n$ converges absolutely for $|x| < R$, and diverges for $|x| > R$, then we say R is the radius of convergence of the power series.

How to find?

Use the ratio / root tests.

$$\lim_{n\to\infty} \left| \frac{a_{n+1} \cdot x^{n+1}}{a_n \cdot x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |x| < 1$$

$$|x| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R.$$

$$\lim_{n\to\infty} |a_n \cdot x^n|^{1/n} = |x| \cdot \lim_{n\to\infty} |a_n|^{1/n} < 1$$

$$|x| < \lim_{n\to\infty} |1/a_n|^{1/n} = R$$

e.g. Find the radius of Convergence for

$$2x + x^2 + 8/9\, x^3 + x^4 + \cdots = \sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$$

sol

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2^n}{n^2} \cdot \frac{(n+1)^2}{2^{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{1}{2} \left| \frac{n+1}{n} \right|^2$$

$$= \frac{1}{2}$$

$$R = \lim_{n\to\infty} \left| 1/a_n \right|^{1/n} = \lim_{n\to\infty} \left| \frac{n^2}{2^n} \right|^{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{2} \left| n^2 \right|^{1/n}$$

$$= \frac{1}{2}$$

Prop $f(x) = \sum_{n=0}^{\infty} a_n x^n$: function defined by a power series and suppose $|x| < R$ where R is the radius of Convergence. Then

- f is continuous, differentiable and integrable.
- $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$
- $\int f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

] term by term differentiation / integration.

e.g We know $1+x+x^2+\dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$

Find a power series expansion of

1) $\log(1-x)$ 2) $\frac{x}{(1-x)^2}$ 3) $\arctan(x)$

1) $\log(1-x) = \int -\frac{1}{(1-x)} = \int \sum_{n=0}^{\infty} -x^n$

$$= \sum_{n=0}^{\infty} -\frac{1}{n+1} x^{n+1}$$

$$= \sum_{n=1}^{\infty} -\frac{1}{n} x^n$$

2) $\frac{x}{(1-x)^2} = x \cdot \left(\frac{1}{(1-x)}\right)' = x \cdot \left(\sum x^n\right)'$

$$= \sum n x^n$$

3) $\arctan x = \int \frac{1}{1+x^2}$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum (-x)^n = \sum (-1)^n \cdot x^n$$

$$\frac{1}{1+x^2} = \sum (-1)^n x^{2n}$$

$$\arctan x = \int \sum (-1)^n x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \square$$

Taylor Series

Taylor series of f at c:

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $f(x) = \sum \frac{f^{(n)}(c)}{n!}(x-c)^n$ for $|x-c| < R$, then we say f is an analytic function near c

Analytic function = Can represent f as a (Taylor) power series

$\Rightarrow$ Can perform term-by-term differentiation / integration

Maclaurin series: Taylor series at 0.

$$f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''}{2}x^2 + \cdots$$

e.g 1) e^x 2) $\sin x$ 3) $\cos x$

1) $e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$

$(e^x)' = 0 + 1 + x + \frac{1}{2!}x^2 + \cdots = \sum \frac{x^n}{n!} = e^x$

$\int e^x = C + x + \frac{1}{2!}x^2 + \cdots = \sum \frac{x^n}{n!} + (C-1) = e^x + D$

2) $\sin' x = \cos x$ $\cos' x = -\sin x.$

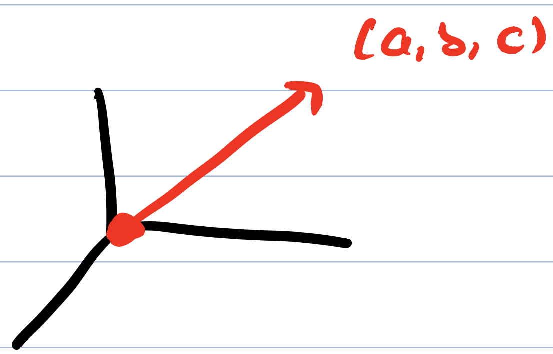
$\sin 0 = 0$ $\cos 0 = 1$

$$\sin x = 0 + 1\cdot x + 0\cdot x^2 + \frac{-1}{3!}x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

3) $$\cos x = 1 + 0\cdot x + \frac{(-1)}{2!}x^2 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Calculus III.

Vectors



$\vec{v} = (a, b, c)$

magnitude / length : $|\vec{v}| = \sqrt{a^2+b^2+c^2}$

direction : $\frac{\vec{v}}{|\vec{v}|}$

$$\vec{v} = \text{magnitude} \cdot \text{direction}$$
$$= \frac{\vec{v}}{|\vec{v}|} \cdot |\vec{v}|$$

ijk-notation : $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$

$i = (1,0,0)$ $\quad j = (0,1,0)$ $\quad k = (0,0,1)$

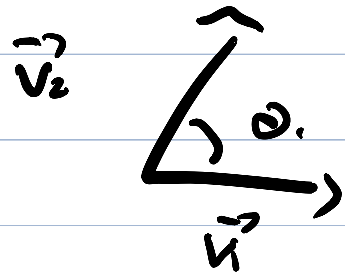
$\vec{v_1} = (a_1, b_1, c_1)$ $\quad \vec{v_2} = (a_2, b_2, c_2)$

dot product : $\vec{v_1} \cdot \vec{v_2} = a_1 a_2 + b_1 b_2 + c_1 c_2$

geometric interpretation :

$$\vec{v_1} \cdot \vec{v_2} = |\vec{v_1}||\vec{v_2}| \cos\theta$$

$$\cos\theta = \frac{\vec{v_1} \cdot \vec{v_2}}{|\vec{v_1}||\vec{v_2}|}$$



Cross product : $\vec{v_1} \times \vec{v_2} = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

$$= (b_1 c_2 - b_2 c_1)\, i + (a_2 c_1 - a_1 c_2)\, j + (a_1 b_2 - a_2 b_1)\, k.$$

geometric interpretation :

$$|\vec{v_1} \times \vec{v_2}| = |\vec{v_1}||\vec{v_2}| \sin\theta.$$